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Entire Solutions of Nonhomogenous Differential Equations

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In this paper, entire solutions of *nonhomogenous* differential equations of second order with polynomial coefficients in complex domain are considered. Bounds on the type and maximum modulus are obtained. Conditions for a solution to be entire and univalent are also given. © 1991 Academic Press, Inc

1. INTRODUCTION

Let

$$L_n(w, P) = P_0(z)\, w^{(n)} + P_1(z)\, w^{(n-1)} + \cdots + P_n(z)\, w = g(z) \tag{1.1}$$

be a differential equation with polynomial coefficients $P_j(z)$, where

$$\deg P_0 \geqslant \deg P_j \qquad (j = 1, 2, ..., n). \tag{1.2}$$

It is known that if g is an entire function of bounded index then entire solutions of (1.1) will also be of bounded index [4].

DEFINITION. An entire function $f\colon \mathbb{C} \to \mathbb{C}$ is said to be of bounded index (b.i.) if there exists an integer $N \geqslant 0$ such that

$$\max_{0 \leqslant j \leqslant N} \frac{|f^{(j)}(z)|}{j!} \geqslant \frac{|f^{(n)}(z)|}{n!} \tag{1.3}$$

for all $z \in \mathbb{C}$ and all $n = 0, 1, 2, ...$. The least such integer N is called the index of f. If f is the b.i. N then it is of exponential type $\leqslant N+1$ [8]. In the case when (1.1) is homogenous, i.e., $g(z) \equiv 0$, and all the P_i are constants, a bound on the index of the f is known [8]. Furthermore for the homogenous equation bounds on the order and the type of the solution in the general case are known [6].

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Suppose the P_i's are polynomials of degree not exceeding d, write $P_j(z) = a_j z^{d_j} +$ lower degree terms, $j = 0, 1, ..., n$, and suppose $a_0 \neq 0$. Note that $a_j = 0$ if the degree of $P_j(z)$ is less than d.

THEOREM A [6]. *Let q be the least nonnegative integer such that*

$$(n+q)!\,|a_0| > \sum_{j=1}^{n} (n+q-j)!\,|a_j|. \tag{1.4}$$

If f is entire and satisfies the equation (1.1) *with $g(z) \equiv 0$ and condition* (1.2) *is satisfied, then*

$$T(f) \equiv \limsup_{r \to \infty} \frac{\log M(r, f)}{r} \leqslant n + q, \tag{1.5}$$

where $M(r, f)$ is the maximum modulus and $T(f)$ is the type of f.

In this paper we give an extension of this theorem to the non-homogenous case.

THEOREM 1. *Let q be as in* (1.4). *Let M be the index of g. If f is entire and satisfies* (1.1) *and condition* (1.2) *is satisfied, then*

$$T(f) \leqslant n + q + \frac{M+1}{(q+n)\cdots(q+1)} \tag{1.6}$$

EXAMPLE 1.1.

$$zw'' + 2w' - zw = e^z$$

has an entire solution $e^z/2$. Here $q = 0$ and $M = 1$, the right hand side of (1.6) is 3, and $T(f)$ is 1. Theorems A and 1 above do not give a bound on $|f(z)|$ for all z. Shah [6] has obtained such a bound for the homogenous case of (1.1). He states his results for the second order differential equation—the type most frequently encountered in mathematical physics—but his results are true for equations of order n.

THEOREM B [6]. *Let an entire function f satisfy the equation*

$$P_0(z)\, w'' + P_1(z)\, w' + P_2(z)\, w = 0, \tag{1.7}$$

where $P_0(z) = a_0 z^2 + \alpha_1 z + \alpha_0$, $P_1(z) = a_1 z^2 + \beta_1 z + \beta_0$,

$$P_2(z) = a_2 z^2 + \delta_1 z + \delta_0 \quad \text{and} \quad a_0 \neq 0. \tag{1.8}$$

Let q be the least nonnegative integer such that

$$(2+q)(1+q)|a_0| > (1+q)|a_1| + |a_2|. \tag{1.9}$$

Then

$$|f(z)| \leqslant A \exp((2+q)|z|) \tag{1.10}$$

for all z. Here A depends on the coefficients of the polynomials P_i. Observe that (1.9) is (1.4) for $n=2$.

We extend this theorem to the nonhomogenous case.

THEOREM 2. *Let an entire function f satisfy the equation*

$$P_0(z)\,w'' + P_1(z)\,w' + P_2(z)\,w = g(z), \tag{1.11}$$

where g is of bounded index. Let q satisfy (1.9). *Then*

$$|f(z)| \leqslant A \exp\left(\left(2+q+\frac{M+1}{(q+2)(q+1)}\right)|z|\right), \tag{1.12}$$

where M is the index of g and A is a function of the coefficients of the P_i's and g and is given in (2.7).

2. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. We may assume that f is not a polynomial and that $N \geqslant n+q$. For $T(f) \leqslant N+1$ and so if $N < n+q$ then $T(f) \leqslant n+q-1+1 = n+q$ and there is nothing to prove. Write

$$\sum_{j=1}^{n} (n+q-j)!\,|a_j|/(n+q)!\,|a_0| = 1-c. \tag{2.1}$$

Choose R such that for $|z| \geqslant R$ we have

$$\frac{1}{|P_0(z)|} < \frac{c}{4} \tag{2.2.1}$$

$$\sum_{j=1}^{n} |P_j(z)|\,\frac{(n+q+1-j)!\,|P_0(z)|}{(n+q+1)!} < \frac{c}{2}$$

$$+\sum_{j=1}^{n} (n+q+1-j)!\,|a_j|/(n+q+1)!\,|a_0| \tag{2.2.2}$$

$$\sum_{t=1}^{n+q+1} \sum_{\substack{i+j=t\\ 0\leqslant j\leqslant n\\ 1\leqslant i\leqslant q+1}} \binom{q+1}{i} \left|\frac{P_j^{(i)}(z)}{P_0(z)}\right| < \frac{c}{4}. \tag{2.2.3}$$

Note that (1.4) and (2.1) imply that $|P_0(z)| > 1$.

Write

$$\omega(s)=\max_{0\leqslant p\leqslant s}\frac{|f^{(p)}(z)|}{p!}+\frac{1}{(n+q)\cdots(q+1)}\operatorname*{Max}_{0\leqslant p\leqslant M}\frac{|g^{(p)}(z)|}{p!}. \qquad (2.3)$$

For notational convenience we will suppress explicit dependence of our functions on z. For example, we will write $f^{(p)}$ instead of $f^{(p)}(z)$.

Differentiate (1.1) with respect to z, q times for $|z|\geqslant R$ (cf. [6])

$$P_0 f^{(n+q)}=-\sum_{t=1}^{n+q}\sum_{\substack{i+j=t\\0\leqslant j\leqslant n\\0\leqslant i\leqslant q}}\binom{q}{i}P_j^{(i)}f^{(n+q-t)},$$

and

$$\frac{|f^{(n+q)}|}{(n+q)!}\leqslant\Sigma_1+\Sigma_2+\frac{1}{|P_0|}\frac{|g^{(q)}|}{q!},$$

where

$$\begin{aligned}\Sigma_1&\leqslant\sum_{j=1}^{n}\frac{|P_j|}{|P_0|}\frac{|f^{(n+q-j)}|}{(n+q)!}\\&\leqslant\left\{\sum_{j=1}^{n}\frac{|P_j|}{|P_0|}\frac{(n+q-j)!}{(n+q)!}\right\}\operatorname*{Max}_{0\leqslant p\leqslant n+q-1}\frac{|f^{(p)}|}{p!}\\&=\left\{\frac{c}{2}+1-c\right\}\\&=\left(1-\frac{c}{2}\right)\max_{0\leqslant p\leqslant n+q-1}\frac{|f^{(p)}|}{p!}\end{aligned}$$

by (2.2.2).

Likewise

$$\begin{aligned}\Sigma_2&\leqslant\left\{\sum_{t=1}^{n+q}\sum_{\substack{i+j=t\\0\leqslant j\leqslant n\\1\leqslant i\leqslant q}}\binom{q}{i}\frac{(n+q-t)!}{(n+q)!}\left|\frac{P_j^{(i)}}{P_0}\right|\right\}\operatorname*{Max}_{0\leqslant p\leqslant n+q-1}\left|\frac{f^{(p)}}{p!}\right|\\&\leqslant\frac{c}{4}\max_{0\leqslant p\leqslant n+q-1}\frac{|f^{(p)}|}{p!}\end{aligned}$$

by (2.2.3).

Finally

$$\frac{1}{|P_0|}\frac{|g^{(q)}|}{(q+n)!}\leqslant\frac{c}{4}\frac{1}{(q+n)\cdots(q+1)}\max_{0\leqslant p\leqslant M}\frac{|g^{(p)}|}{p!}$$

by (2.2.1).

Therefore

$$\frac{|f^{(n+q)}|}{(n+q)!} \leqslant \left(1-\frac{c}{2}\right) \max_{0 \leqslant p \leqslant n+q-1} \frac{|f^{(p)}|}{p!} + \frac{c}{4} \max_{0 \leqslant p \leqslant n+q-1} \frac{|f^{(p)}|}{p!}$$

$$+\frac{c}{4(q+n)\cdots(q+1)} \leqslant \omega(n+q-1) \tag{2.4}$$

by (2.3).

Let $\alpha \in C$, $|\alpha| = 1$, α fixed, and for $x \geqslant R$, write

$$G(x) = \max_{0 \leqslant j \leqslant n+q-1} \frac{|f^{(j)}(\alpha x)|}{j!}$$

$$+\frac{1}{(n+q)\cdots(q+1)} \max_{0 \leqslant p \leqslant M} \frac{|g^{(p)}(\alpha x)|}{p!}. \tag{2.5}$$

G is continuous and piecewise continuously differentiable and not zero. Hence for all x, except possibly for a set of measure zero,

$$G'(x) \leqslant (n+q) \max_{0 \leqslant j \leqslant n+q} \frac{|f^{(j)}(\alpha x)|}{j!}$$

$$+\frac{M+1}{(n+q)\cdots(n+1)} \max_{0 \leqslant p \leqslant M+1} \frac{|g^{(j)}(\alpha x)|}{p!}.$$

Using the fact that g is of bounded index and (2.4) we get

$$G'(x) \leqslant \left(n+q+\frac{M+1}{(n+q)\cdots(n+1)}\right) G(x).$$

Thus

$$G(x) \leqslant G(x_0) \exp\left\{\left(n+q+\frac{M+1}{(n+q)\cdots(q+1)}\right)(x-x_0)\right\},$$

where x_0 is a quantity $\geqslant R$.

Write $\alpha x = z$ and we have for $|z| > R$,

$$|f(z)| \leqslant A \exp\left\{\left(n+q+\frac{M+1}{(n+q)\cdots(q+1)}\right)|z|\right\},$$

where $A = \exp(-x_0)\, G(x_0)$, x_0 is a constant $> R$, and $G(x_0)$ is defined in (2.5).

Note 1. When $g(z) \equiv 0$, i.e., when the equation is homogenous, we have $M = 0$. In this case (1.6) is almost as good as (1.5) except for the additional term of $1/(q+n) \cdots (q+1)$.

Proof of Theorem 2. The proof is a variation of that given in [6]. Write

$$\{(1+q)|a_1| + |a_2|\}/(2+q)(1+q)|a_0| = 1 - c,$$

where $c > 0$. Differentiate (1.11) q times with w replaced by f, and let $R \geqslant 1$ be such that $|P_0(z)| \geqslant 1$ for $|z| \geqslant R$. Suppose first that $q \geqslant 2$. Then for $|z| \geqslant R$,

$$\frac{|f^{(q+2)}|}{(q+2)!} \leqslant \frac{|f^{(q+1)}|}{(q+1)!} |\Sigma_1| + \frac{|f^{(q)}|}{q!} |\Sigma_2| + \frac{|f^{(q-1)}|}{(q-1)!} |\Sigma_3| + \frac{|f^{(q-2)}|}{(q-2)!} |\Sigma_4| + \frac{|g^{(q)}|}{|P_0|(q+2)!}, \tag{2.6}$$

where the Σ_i are defined as in [6]. They are given in terms of the P_i's and their first and second derivatives. One can estimate Σ_i in terms of the coefficients of the polynomials P_0, P_1, P_2 and show that for sufficiently large R [see 6]

$$\frac{|f^{(q+2)}|}{(q+2)!} \leqslant \max_{0 \leqslant m \leqslant q+1} \frac{|f^{(m)}|}{m!} + \frac{1}{(q+1)(q+2)} \frac{|g^{(q)}|}{(q+2)!}. \tag{2.7}$$

We have used (2.6) and stress again that R depends on the coefficients of the P_i's. It can be easily shown that (2.7) is also valid for $q = 0$ and $q = 1$.

Write

$$\Omega(s) = \max_{0 \leqslant p \leqslant s} \frac{|f^{(p)}(z)|}{p!} + \frac{1}{(q+1)(q+2)} \max_{0 \leqslant p \leqslant M} \frac{|g^{(p)}|}{p!}.$$

Then as in the proof of Theorem 1, for $|z| \geqslant R$,

$$\frac{|f(q+2)|}{(q+2)!} \leqslant \Omega(q+1).$$

Define $G(x)$ as in Theorem 1. A similar reasoning gives

$$G'(x) \leqslant \left(q + 2 + \frac{M+1}{(q+2)(q+1)}\right) G(x).$$

So,

$$|f(z)| \leqslant G(R) \exp\left(q + 2 + \frac{M+1}{(q+2)(q+1)}\right) (|z| - R) \qquad \text{for} \quad |z| \geqslant R.$$

Observe that

$$G(R)=\max_{0\leqslant j\leqslant q+1}\frac{M(R,f^{(j)})}{j!}+\frac{1}{(q+1)(q+2)}\max_{0\leqslant p\leqslant M}\frac{M(R,g^{(p)})}{p!}.$$

Thus we can take

$$A=\begin{cases}\exp\left\{-(q+2)-\dfrac{M+1}{(q+2)(q+1)}\right\}R\left[\max\limits_{0\leqslant j\leqslant q+1}\dfrac{M(R,f^{(j)})}{j!}\right.\\ \qquad\left.+\dfrac{1}{(q+2)(q+1)}\max\limits_{0\leqslant p\leqslant M}\dfrac{M(R,g^{(p)})}{p!}\right] & \text{for } |z|\geqslant R\\ M(R,f) & \text{for } |z|<R,\end{cases}$$

where R depends on the coefficients P_i and on c. This theorem can be generalized to equations of order n. However, determination of R in the general case can be quite complicated.

3. Some Sufficient Conditions for the Entire and Univalent Solutions

In this section we will explore conditions under which a solution of a nonhomogeneous equation is entire. We will also state conditions under which the solution and its derivatives are all univalent in the unit disc. We will restrict our attention to second order differential equations (DE) of a special form. Not much generality is lost by these restrictions for many second order equations are of this form and extension to the nth order equations is readily done.

The DE we will be studying is (cf. [7])

$$z^2w''+(\beta_0z^2+\beta_1z)\,w'+(\gamma_0z^2+\gamma_1z+\gamma_2)\,w=g(z), \tag{3.1}$$

where g is an entire function.

Theorem 3 below gives conditions under which this equation has an entire solution. Part (a) is a result on a general second order DE with regular singular point at the origin. Surely this must be a known result but since we are unable to find a reference for it we state and prove it. It is obviously true under more general conditions but we state it only in the form we need.

THEOREM 3. (a) *Let*

$$z^2w''+zp(z)\,w'+q(z)\,w=g(z), \tag{3.2}$$

where $p(z)=\sum_{n=0}^{\infty} p_n, z^n$, $q(z)=\sum_{n=0}^{\infty} q_n z^n$, $g(z)=z^m \sum_{n=0}^{\infty} b_n z^n$, $b_0 \neq 0$ *are entire functions. Assume that the indicial equation*

$$I(s) \equiv s(s-1)+p_0 s+q_0=0 \tag{3.3}$$

has a nonnegative integer solution v *such that* $I(v+n) \neq 0$ *for* $n \geqslant 1$. *Further assume that* m *is an integer larger than both of the roots of the indicial equation. Then* (3.2) *has an entire solution of the form*

$$f(z)=z^\alpha \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0, \tag{3.4}$$

where α *can be chosen to be equal to either* v *or* m.

(b) *Assume* $\gamma_2=0$ *and* $\beta_1>0$ *and* $g(z)=z\sum_{n=0}^{\infty} b_n z^n$, $b_0 \neq 0$ *in* (3.1). *Then* (3.1) *has an entire solution of the form*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0. \tag{3.5}$$

(c) *Assume* $\beta_1+\gamma_2=0$, $\beta_1>-2$, $g(z)=z^2\sum_{n=0}^{\infty} b_n z^n$, $b_0 \neq 0$ in (3.1). *Then* (3.1) *has a solution of the form*

$$F(z)=\sum_{n=1}^{\infty} a_n z^n, \qquad a_1 \neq 0. \tag{3.6}$$

Note 2. The fact that some kind of condition on m is necessary is illustrated by the equation $z^2 w''=1$ all of whose solutions are logarithmic. Note also that Theorem 3 is an existential result; it says only that solutions of a certain form exist.

Proof. (a) The homogenous equation corresponding to (3.2) has an entire solution (complementary solution) of the form $z^v \sum_{n=0}^{\infty} d_n z^n$ where $d_0 \neq 0$ is arbitrary. This result is standard (see, e.g., [2]). To find a particular series solution to (3.2) let $w=z^m \sum_{n=0}^{\infty} c_n z^n$ and substitute it in (3.2). Equating the coefficients of the terms of lowest degree (i.e., in this case the coefficients of z^m) on both sides of the equation, we obtain $\{m(m-1)+p_0 m+q_0\}\, c_0=b_0$ or $I(m)\, c_0=b_0$. Since $I(m) \neq 0$, this equation can be solved for c_0. Furthermore $c_0 \neq 0$ since $b_0 \neq 0$. The recurrence equation for c_n $(n \geqslant 1)$ is

$$I(n+m)\, c_n=-\sum_{k=0}^{n-1} [(m+k)\, p_{n-k}+q_{n-k}]\, c_k+b_n. \tag{3.7}$$

By assumption $I(n+m) \neq 0$ so these equations can be solved for c_n. So a formal particular solution exists. We have to show that this series defines an entire function. The proof is similar to the standard majorization proof for homogenous equations [9]. However, we sketch it here for the sake of completeness. Let $r>0$ be arbitrary. By analyticity there is a constant M such that

$$|p_n| \leqslant \frac{M}{r^n}, \qquad |q_n| \leqslant \frac{M}{r^n}, \qquad |b_n| \leqslant \frac{M}{r^n}, \text{ for } n \geqslant 0. \tag{3.8}$$

Let μ be the other root of indicial equation. Since $I(n+m)=(n+m-\nu)(n+m-\mu)$, $I(n+m)>0$ by hypothesis for $n \geqslant 1$. From (3.7) and (3.8) we have

$$|c_n|\, I(n+m) \leqslant M \sum_{k=0}^{n-1} |c_k| \frac{m+k+1}{r^{n-k}} + \frac{M}{r^n} \qquad \text{for} \quad n \geqslant 1. \tag{3.9}$$

Define $C_0 = |c_0|$, and C_n inductively by the expression on the right of (3.9) divided by $I(n+m)$ with $|c_k|$ replaced by C_k. We claim that $C_n \geqslant |c_n|$. This can be readily proved by induction. After multiplying C_{n-1} by $1/r$ and subtracting it from C_n we get for $n \geqslant 1$ that

$$I(n+m)\, C_n - I(n+m-1) \frac{C_{n-1}}{r} = M(m+n) \frac{C_{n-1}}{r} + \frac{M}{r^{n+1}} - \frac{M}{r^{n+1}}.$$

Thus $\lim_{n \to \infty} (C_n/C_{n-1}) = 1/r$ and since $C_n \geqslant |c_n|$, the radius of convergence of w is at least r. Since r is arbitrary, w is entire. Now this particular solution will give (3.4) with $\alpha = m$ and when added to the complementary solution it will give (3.4) with $\alpha = \nu$.

(b) In this case the indicial equation has two solutions $s=0$ and $s=1-\beta_1$, and $m=1$. So the hypotheses of part (a) are satisfied and we have a solution of the form (3.5) by taking $\alpha=\nu=0$ in (3.4).

(c) The solutions to the indicial equation are $s=1$ and $s=-\beta_1$, and $m=2$. Again the hypotheses of part (a) are satisfied and there is a solution of form (3.6) by taking $\alpha=\nu=1$ in (3.4).

Univalent solutions of second order differential equations have been the subject of study (see the references in [7]). In [7] Shah shows that if

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 = 1$$

and

$$n \left| \frac{a_n}{a_{n-1}} \right| \leqslant \log 2$$

for all n, $n \geqslant 2$ (log is in base e) then f and all of its derivatives are starlike univalent in the unit disc. He calls the set of functions with this property class **E**. (This class is a subset of extensively studied class E with the property that the functions in this class are univalent together with their derivatives [see 10].) In addition the type of a function in this class can not be larger than 1/2. He then states conditions to be satisfied by coefficients of P_i in (3.1) so that the entire solution of the homogeneous equation corresponding to (3.1) ($g(z) \equiv 0$) will be in class **E**. In Theorem 4 we present analogous results for the nonhomogenous case. In this theorem we are looking for solutions $f(z)$ in which $f(z)$ and all of its derivatives are univalent, thus we are only interested in cases where the indicial equation has solution $s=0$ or $s=1$ similar to situations in parts (b) and (c) of Theorem 3.

THEOREM 4. *Assume that the notation and hypotheses of Theorem* 3(b) *and* (c) *hold as needed for each of the cases below. Write*

$$\begin{matrix} K(f) \\ k(f) \end{matrix} = \begin{matrix} \sup \\ \inf \end{matrix} \left\{ \left| \frac{f^{(n)}(0)}{f^{(n-1)}(0)} \right| \right\}, \qquad n \geqslant 1$$

for $f(z)$ as defined in (3.5). *Assume that $K(F)$ and $k(F)$ are similarly defined for $F(z)$ in* (3.6) *with $n \geqslant 2$. Also write*

$$\tau = \sup \left| \frac{b_n}{(n-1)\, n a_n} \right|, \qquad n \geqslant 2.$$

(a) *If $K(f) \leqslant \log 2$ then $(f-a_0)/a_1$ is in* **E**, *and if $K(F) \leqslant \log 2$ then F/a_1 belongs to* **E**.

(b) *If* $\gamma_2 = 0, \beta_1 > 0, \quad \gamma_1 \neq 0, k(f) > 0, \quad |\beta_0| + |\gamma_0|/k(f) + K(f)\,\tau \leqslant \log 2$, *and* $|\gamma_1| \leqslant \beta_1 \log 2$, *then* $(f-a_0)/a_1$ is in **E**.

(c) *If* $\beta_1 + \gamma_2 = 0, \quad \beta_1 > 0, \quad |\beta_0| + \gamma_0/k(F) + K(F)\,\tau \leqslant \log 2, \quad |\gamma_1| \leqslant (\beta_1 \log 2)/2$ *then F/a_1 is in* **E**.

Proof. (a) This is proved readily for

$$|f^{(n)}(0)/f^{(n-1)}(0)| = n\, |a_n/a_{n-1}| \leqslant K(f) \leqslant \log 2.$$

We just have to note that $a_0 \neq 0$ by assumption. A similar argument works for F.

(b) The recurrence relation for a_n which we obtain from substituting (3.5) in (3.1) is

$$a_n = -a_{n-1} \frac{\beta_0(n-1) + \gamma_1}{n(n-1+\beta_1) + \gamma_2} - \gamma_0 \frac{a_{n-2}}{n(n-1-\beta_1) + \gamma_2} + \frac{b_n}{n(n-1+\beta_1) + \gamma_2}. \tag{3.10}$$

Thus

$$n\,|a_n/a_{n-1}| \leqslant n\left\{|\beta_0|(n-1)+|\gamma_1|+|\gamma_0|\frac{(n-1)}{k(f)}+|b_n/a_{n-1}|\right\}\Big/$$
$$\{n(n-1+\beta_1)+\gamma_2\}$$
$$\leqslant \log 2 \text{ for } n \geqslant 2.$$

That is because

$$|b_n/a_{n-1}|=(n-1)|b_n/n(n-1)|\,|na_n/a_{n-1}| \leqslant (n-1)\,\tau K(f),$$

and $\gamma_2=0$. Now since $a_1=-\gamma_1 a_0/\beta_1 \neq 0$ the conclusion follows.

(c) Substituting $\gamma_2=-\beta_1$ in (3.10) we have

$$n\left|\frac{a_n}{a_{n-1}}\right|$$
$$\leqslant \frac{n}{(n-1)(n+\beta_1)}\left\{(n-1)\,\beta_0+|\gamma_1|+|\gamma_0|\left|\frac{a_{n-2}}{a_{n-1}}\right|+\left|\frac{b_n}{a_{n-1}}\right|\right\}$$
$$\leqslant \left\{(n-1)\,\beta_0+|\gamma_1|+(n-1)\left|\frac{\gamma_0}{k(F)}\right|+K(f)\,\tau(n-1)\right\}\frac{n}{(n-1)(n+\beta_1)}$$
$$\leqslant \frac{(n-1+(1/2)\,\beta_1)}{(n-1)(n+\beta_1)} \leqslant 1 \qquad \text{for} \quad n \geqslant 2.$$

We have assumed that $a_0=0$ above. Since $a_1 \neq 0$, the conclusion follows.

4. Examples

4.1. Consider the Coulomb equation with the "forcing function" $z^m e^z$:

$$z^2 w''+\{z^2-2\eta z-L(L+1)\}\,w=z^m e^z. \tag{4.1}$$

The indicial equation has two roots $-L$ and $L+1$. Thus according to Theorem 3 if L is a nonnegative integer and m is greater than $L+1$ then there is an entire solution to this equation. To illustrate Theorems 1 and 2 for this entire solution which is also of bounded index let $L=0$, $\eta=1$, and $m=2$. Differentiate (4.1) once, eliminate e^z, and assume that $R=4$:

$$z^2 w'''-z^2 w''+(z^2-2z)\,w'+(2z-z^2+2)\,w=0.$$

Thus,

$$|w'''| \leqslant \left\{4+\frac{4}{|z|}+\frac{4}{|z|^2}\right\} \max\left\{\frac{|w''|}{2!}, \frac{|w'|}{1!}, |w|\right\}.$$

Since $|z| \geqslant R$, we have

$$\frac{|w'''|}{3!} \leqslant \max\left\{\frac{|w''|}{2!}, \frac{|w'|}{1!}, |w|\right\} \qquad \text{for} \quad |z| \geqslant R.$$

Using the technique which leads to derivation of (2.7) (see [6]) one can show that likewise for all $k \geqslant 0$

$$\frac{|w^{(k)}(z)|}{k!} \leqslant \max\left\{\frac{|w''|}{2!}, \frac{|w'|}{1!}, |w|\right\}.$$

Here $q=0$, $M=1$. An upper bound on the index of $e^z z^2$ is 5 (see Example 4.3 below). Thus Theorems 1 and 2 applied to Eq. (4.1) yield $T(f) \leqslant 5$ and $|f(z)| \leqslant A \exp(5|z|)$ where A is defined by

$$A=\begin{cases}\exp(-5R)[\max\{M(R,f), M(R,f'), M(R,f'')/2\}+\frac{1}{2}e^R] & \text{for } |z| \geqslant R\\ M(R,f) & \text{for } |z|<R,\end{cases}$$

where $R=4$.

4.2. Consider the nonhomogenous Bessel equation

$$z^2w''+zw'+(z^2-v^2)w=z^\rho g(z),$$

where v is a nonnegative integer and $g(z)$ is entire. If ρ is an integer larger than v then by Theorem 3(a) we will have an entire solution. One of the interesting cases is when $g(z) \equiv 1$ and ρ is an integer larger than v [1]. To use Theorem 3(a), ρ must be an integer greater than v. The explicit solution is given by

$$f(z)=2^{\rho-2}\Gamma\left(\frac{\rho+v}{2}\right)\Gamma\left(\frac{\rho-v}{2}\right) \times \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\rho+2k}}{\Gamma((\rho+v)/2+k+1)\,\Gamma((\rho-v)/2+k+1)}.$$

This an entire function of order 1 and type 1. The exponent in Theorem 2 which is also the constant in the right hand side of (1.6) is given below. Notice that the minimum q satisfying (1.9) is 0 and the index of g is ρ.

$$n+1+\frac{M+1}{(q+2)(q+1)}=2+\frac{\rho+1}{2}. \tag{4.2}$$

A related case, when $\rho = \nu + 1$ and $g(z) = 4(z/2)^{\nu+1}/\sqrt{\pi}\,\Gamma(\nu + 1/2)$, is of special interest. In this case the series solution is Struve's function [1, 11]

$$H_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{\nu+2k+1}}{\Gamma(k+3/2)\,\Gamma(\nu+k+3/2)}.$$

This function is order and type 1. The constant corresponding to (4.2) is $2 + (\nu + 2)/2$.

4.3. Consider the nonhomogenous differential equation

$$z^2 w'' - (2z^2 - z)\,w' + (2z^2 - z - \nu^2)\,w = e^z z^\rho, \tag{4.3}$$

where as in the foregoing example ρ is an integer greater than ν. This equation is closely related to Bessel's equation [1]. In fact its series solution is e^z times the series for $f(z)$ given in Example 4.2. To compute the quantity $n + (M+1)/(q+2)(q+1)$ we need to find the index M of $e^z z^\rho$. We look at two cases. Let $\rho = 1$. Since $\nu \geqslant 0$ and is less than ρ, $\nu = 0$ and we have the equation

$$z^2 w'' - (2z^2 - z)\,w'' + (2z^2 - z)\,w = e^z z.$$

If $g(z) = e^z z$, $g^{(n)}(z)/n! = |e^z|\,|z+n|/n!$. It can be easily shown that $|z| \geqslant |z+n|/n!$ for $n \geqslant 4$ and $|z| > 4/23$. Also $|z+1| \geqslant |z+n|/n!$ for $n \geqslant 4$ and $|z| < 4/5$. Thus the index of $g(z)$ is bounded above by 4. Notice that the minimum q satisfying (1.9) is 2. Therefore $4 + (M+1)/12 \leqslant 53/12$. The other case of interest is $\rho = 2$. In this case ν can be either 0 or 1. Obviously the more interesting case is $\nu = 1$. If $g(z) = e^z z^2$ then $g^{(n)}(z)/n! = |e^z|\,|z^2 + 2nz + n(n-1)|/n!$. One can show that $|z^2|$ is larger than this quantity for $|z| > 1/4$ and $n \geqslant 5$. Also this same expression is less than $|z^2 + 2z|$ for $n \geqslant 5$ and $|z| \leqslant 1/4$. Thus the upper bound on the index of the $g(z)$ is 5 and $4 + (M+1)/12 \leqslant 9/2$.

4.4. Consider

$$z^2 w'' + azw' - a^2 z^2 w = a^2 z e^{az}, \qquad a > 0.$$

$f(z) = e^{az}$ is a solution of this equation. Here we have $\gamma_2 = 0$, $s = 0$, and $s = 1 - a$. We have the situation of Theorem 3(b). Observe that

$$n \left| \frac{a_n}{a_{n-1}} \right| = a.$$

Thus if $a \leqslant \log 2$ then the normalized function $f_0 = (f-1)/a$ is in class **E**. However, if $\log 2 < a \leqslant \pi$ then f_0 is not in **E** but is in class E. For $a > \pi$, f_0 is not univalent.

4.5. Consider

$$z^2w'' + \pi z w' - \pi^2 z^2 w = z\{\pi^2 e^{\pi z} - \pi^2 b z^3 - \pi^2 a z^2 + 2bz(\pi + 1)\}.$$

This equation has a solution of the form $e^{\pi z} + az + bz^2$ for arbitrary constants a and b. However, since the type of this function is larger than 1/2 it can't be in class **E**. Nonetheless Lachance [5] has shown that for $a = \pi e^{-\pi}/35$ and $b = 18$ this function is in class E. Also note that the equation is of the form Theorem 3(b).

4.6. Consider

$$2z^2 w'' + (z^2 - z)\, w' + w = z^2.$$

Here $\beta_1 + \gamma_2 = 0$, $s = 1/2$, and $s = 1$ are solutions of the indicial equation. A particular solution for the equation is z. By the Frobenius (series) method there is an entire solution for the homogenous equation of the form (see [7])

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{1.3.5...(2n+1)}.$$

Thus a solution of the equation is $h(z) = f(z) + z$. The normalized function $h/2$ has the property that $n\,|a_n/a_{n-1}| \leqslant \log 2$. Thus h is in class **E**.

4.7. Consider

$$z^2 w'' - \frac{z}{2} w' + \left(\frac{z}{4} + \frac{1}{2}\right) w = \frac{z^2}{4}.$$

A solution of the homogenous equation is $\sqrt{z} \sin \sqrt{z}$. This equation is of the form Theorem 3(c). A particular solution is z. It is easily seen that the sum of these two functions after normalization is also in **E** (cf. [7]).

4.8. Consider the nonhomogenous confluent hypergeometric equation

$$z^2 w'' + z(c - z)\, w' - zaw = z^\sigma e^{\rho z}, \tag{4.4}$$

where $c \geqslant a > 0$. Roots of the indicial equation are 0 and $1 - c$. According to Theorem 3(a) if $\sigma = m$ is a positive integer then there is an entire solution of the equation. When $\rho = 0$ a particular solution of the equation is the nonhomogenous confluent hypergeometric function defined by [11]

$$\begin{aligned} \theta_m(a, c; z) = {} & \frac{(m-1)!\,\Gamma(m+c-1)\,\Gamma(a)}{\Gamma(m+a)\,\Gamma(c)} \\ & \times \sum_{n=0}^{\infty} \frac{\Gamma(m+n+a)\,\Gamma(c)}{\Gamma(a)\,\Gamma(m+n+c)} \frac{z^{n+m}}{(n+m)!}. \end{aligned} \tag{4.5}$$

Obviously when $m > 1$ the function is not univalent. However, the function is m-valently starlike in the unit disc for it satisfies the inequality

$$\sum_{n=m+1}^{\infty} n\,|a_n| \leqslant m\,|a_m|.$$

When $m=1$ the function is univalent but it can not be in class **E** because its type is 1. However, it is easy to show that the normalized function $F(z) = c\theta_1\,(a, c; z)$ is in class E. That is because of Alexander's theorem which stipulates that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $1 = a_1 \geqslant 2a_2 \geqslant \cdots \geqslant na_n \geqslant \ldots$, $a_n \geqslant 0$, then $f(z)$ is close-to-convex. This is readily seen to be true for $F(z)$ and its derivatives. When $\rho \neq 0$ and $\sigma = m$ is a positive integer then the series for a particular solution, still called the nonhomogenous confluent function, is given by [11]

$$A_{\rho,m}(a, c; z) = z^m \sum_{n=0}^{\infty} \frac{\Gamma(m+a+n)\,\Gamma(m)\,\Gamma(m+c-1)}{\Gamma(m+a)\,\Gamma(m+n+1)\,\Gamma(m+c+n)} \times F_{(n+1)}[m, m+c-1, m+a; \rho]\, z^n,$$

where $F_{(n+1)}[m, m+c-1, m+a; \rho]$ stands for the first $n+1$ terms of the hypergeometric series. This is an entire function of bounded index. Using Theorem 1, we can estimate the index of this function.

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